Theorem 11

Change of integration of Riemann Integr

Let f be continuous on rectangle $[a,b]\times[c,d]$

If $g \in R$ on $[a,b]$ and if $h \in R$ on $[c,d]$, then we have.

$$\int_a^b \left\{ \int_c^d g(x)\, h(y)\, f(x,y)\,dy \right\} dx = \int_c^d \left\{ \int_a^b g(x)\, h(y)\, f(x,y)\,dx \right\} dy$$

Proof: Given, (i) f is continuous on $[a,b]\times[c,d]$

(ii) $g \in R$ on $[a,b]$

(iii) $h \in R$ on $[c,d]$

Let $\alpha(x) = \int_a^x g(u)\,du \longrightarrow (1)$, $\forall\ x \in [a,b]$

and $\beta(y) = \int_c^y h(v)\,dv \longrightarrow (2)\ \forall\ y \in [c,d]$

By I^{st} Fundamental theorem on Integral calculus,

Since $g \in R$ on $[a,b]$ and by (1)

$$\alpha'(x) = g(x)$$

Since $h \in R$ on $[c,d]$ and by (2)

$$\beta'(y) = h(y)$$

Since $g \in R$ on $[a,b]$ and $\alpha(x) = \int_a^b g(u)\,du$, $\forall\ x \in [a,$

α is of b.v on $[a,b]$

Since $h \in R$ on $[c,d]$ and $\beta(y) = \int_c^y h(v)\,dv$, $\forall\ y \in [c$

β is of b.v on $[c,d]$

Define, $F(y) = \int_a^b f(x,y)\,d\alpha(x)$

then $F(y) = \int_a^b f(x,y)\,\alpha'(x)\,dx$

$$F(y) = \int_a^b f(x,y)\, g(x)\, dx$$

Also define $G(x) = \int_c^d f(x,y)\, d\beta(y)$

then $G(x) = \int_c^d f(x,y)\, \beta'(y)\, dy$

i.e) $G(x) = \int_c^d f(x,y)\, h(y)\, dy$

then by thm: 16

$G \in R$ on $[a,b]$ and $F \in R$ on $[c,d]$

Also, $\int_a^b G(x)\, d\alpha(x) = \int_c^d F(y)\, d\beta(y)$

$$\Rightarrow \int_a^b G(x)\, \alpha'(x)\, dx = \int_c^d F(y)\, \beta'(y)\, dy$$

$$\Rightarrow \int_a^b G(x)\, g(x)\, dx = \int_c^d F(y)\, h(y)\, dy$$

$$\Rightarrow \int_a^b \left\{ \int_c^d f(x,y)\, h(y)\, g(x)\, dy \right\} dx = \int_c^d \left\{ \int_a^b f(x,y)\, g(x)\, h(y)\, dx \right\} dy$$

Hence the proof.

Section : 7.20

Theorem : 18

Second Fundamental theorem of Integral Calculus.

Assume that $f \in R$ on $[a,b]$ and g be a function defined on $[a,b]$ such that the derivative g' exists in (a,b) and has the value.

$$g'(x) = f(x), \quad \forall\, x \in [a,b]$$

At the end points assume that $g(a+)$ and $g(b-)$ exist and satisfy.

$$g(a) - g(a+) = g(b) - g(b-)$$

then we have, $\int_a^b f(x)\, dx = \int_a^b g'(x)\, dx = g(b) - g(a)$

Proof: Given.

(i) $f \in R$ on $[a,b]$

(ii) g is a function defined on $[a,b]$

Such that g' exists in (a,b) and

$$g'(x) = f(x), \quad \forall \; x \in [a,b]$$

Prove that: $\int_a^b f(x)\,dx = \int_a^b g'(x)\,dx = g(b) - g(a)$

Since $f \in R$ on $[a,b]$ for any given $\epsilon > 0$ there exists a partition P_ϵ such that $\forall \; P \supseteq P_\epsilon$ and $\forall$ choice of t_k in $[x_{k-1}, x_k]$

$$\left| \sum_{k=1}^{n} f(t_k)\,\Delta x_k - \int_a^b f(x)\,dx \right| < \epsilon \longrightarrow (1)$$

Now for every partition,

$P = \{a = x_0, x_1, x_2, \ldots x_n = b\}$ we can write

$$g(b) - g(a) = \sum_{k=1}^{n} [g(x_k) - g(x_{k-1})] \longrightarrow (2)$$

Since g is continuous on $[a,b]$ and g' exists in (a,b) by mean value theorem, $\exists \; t_k \in [x_{k-1}, x_k]$ such that,

$$g'(t_k) = \frac{g(x_k) - g(x_{k-1})}{x_k - x_{k-1}}$$

$$\Rightarrow g'(t_k) = \frac{g(x_k) - g(x_{k-1})}{\Delta x_k}$$

$$\Rightarrow g(x_k) - g(x_{k-1}) = g'(t_k)\,\Delta x_k$$

Using this in (2) we get,

$$g(b) - g(a) = \sum_{k=1} g'(t_k)\,\Delta x_k \quad [\because g'(x) = f(x)]$$

$$\Rightarrow g(b) - g(a) = \sum_{k=1} f(t_k)\,\Delta x_k$$

$\Rightarrow \left| g(b) - g(a) - \int_a^b f\,dx \right| = \left| \sum_{k=1}^{n} f(t_k)\,\Delta x_k - \int_a^b f\,dx \right| < \epsilon$

$\Rightarrow \left| g(b) - g(a) - \int_a^b f\,dx \right| < \epsilon$

Since $\epsilon > 0$ is arbitrary

$g(b) - g(a) - \int_a^b f\,da = 0$

$\Rightarrow g(b) - g(a) = \int_a^b f(x)\,dx$

$= \int_a^b g'(x)\,dx$

Hence $\int_a^b f(x)\,dx = \int_a^b g'(x)\,dx = g(b) - g(a)$ [Given, $f(x) = g'(x)$]

Theorem: 19

Assume $f \in R$ on $[a,b]$. Let α be a function which is continuous on $[a,b]$ whose derivative α' is Riemann integrable on $[a,b]$, then the following integrals exist and are equal such that,

$$\int_a^b f(x)\,d\alpha(x) = \int_a^b f(x)\,\alpha'(x)\,dx$$

proof: Given, (i) $f \in R$ on $[a,b]$

(ii) α is continuous on $[a,b]$

(iii) α' is Riemann Integrable on $[a,b]$

To prove: $\int_a^b f(x)\,d\alpha(x) = \int_a^b f(x)\,\alpha'(x)\,dx$

By the second fundamental theorem of integral Calculus, we have,

$$\int_a^x \alpha'(t)\,dt = \alpha(x) - \alpha(a) \quad \forall\, x \in [a,b]$$

Also we have if $f \in R$ on $[a,b]$ and $g \in R$ on $[a,b]$

and $G(x) = \int_a^x g(t)\,dt \quad \forall\, x \in [a,b]$

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Then $\int_a^b f(x)\, g(x)\, dx = \int_a^b f(x)\, d\,\varphi(x)$

$$= \int_a^b f(x)\, d\,[\alpha(x) - \alpha(a)]$$

$\int_a^b f(x)\, \alpha'(x)\, dx = \int_a^b f(x)\, d\alpha(x)$ [$\alpha(a)$ is constant

Hence the proof

Unit IV

Infinite Series and Infinite product

Def: 8.17 Absolute and Conditional convergence:

A Series $\sum a_n$ is called absolutely convergent if $\sum |a_n|$ converges. It is called conditionally convergent if $\sum a_n$ converges but $\sum |a_n|$ diverges.

Theorem: 8.18

(1) Absolute convergence of $\sum a_n$ implies convergence

Proof:

Hypothesis: $\sum a_n$ is absolute convergent $\longrightarrow$ ①

(ie) $\sum |a_n|$ is convergent

Claim: $\sum a_n$ is convergent.

(ie) TP that the sequence of partial sum of $\sum a_n$ is convergent.

TPT: If $S_n = a_1 + a_2 + \cdots + a_n$, $n = 1, 2$

Then $\{S_n\}$ is convergent

(ie) TPT: $\{S_n\}$ is cauchy

Let $t_n = |a_1| + |a_2| + \cdots + |a_n|$, $n = 1, 2$

(ie) t_n is the n^{th} partial sums of $\sum |a_n|$

$\therefore$ ① $\Rightarrow$ $\{t_n\}$ is convergent

$\Rightarrow$ $\{t_n\}$ is cauchy

given $\epsilon > 0$, $\exists$ $N \in \mathbb{I}$ such that

$$|t_m - t_n| < \epsilon, \ (m, n \geq N) \longrightarrow ①$$

Let $m > n$

Consider,

$$|S_m - S_n| = |a_1 + a_2 + \cdots + a_n + a_{n+1} + \cdots + a_m - (a_1 + a_2 + \cdots a_n)|$$

$$= |a_{n+1} + a_{n+2} + \cdots + a_m|$$

$$|S_m - S_n| \leq |a_{n+1}| + |a_{n+2}| + \cdots + |a_m| \longrightarrow ③$$

Consider.

$$|t_m - t_n| = \Big| |a_1| + |a_2| + \cdots + |a_n| + |a_{n+1}| + \cdots + |a_m|$$
$$- (|a_1| + |a_2| + \cdots + |a_n|)\Big|$$
$$= \Big| |a_{n+1}| + |a_{n+2}| + \cdots + |a_m| \Big|$$
$$= |a_{n+1}| + |a_{n+2}| + \cdots + |a_m| \longrightarrow ④$$

Sub ④ in ③

$$|S_m - S_n| = |t_m - t_n|$$
$$< \epsilon \quad \text{by ①}$$

$|z_1 + z_2| \leq |z_1| + |z_2|$

$||z_1| + |z_2|| = |z_1| + |z_2|$

Thus we have

$$|S_m - S_n| < \epsilon, \ (m, n \geq N)$$

$\Rightarrow$ $\{S_n\}$ is cauchy

Hence the theorem

Note: $\sum (-1)^{n+1} a_n$ or $\sum (-1)^n a_n$ be an alternating series

Suppose $\sum a_n = a_1 - a_2 + a_3 - a_4 + \cdots$

$$P_n = a_1 + a_3 + a_5 + \cdots$$

$$q_n = a_2 + a_4 + a_6 + \cdots$$

$$p_n = \begin{cases} a_n & \text{if } \dots \\ 0 & \text{if } a_n \le 0 \end{cases}$$

$$q_n = \begin{cases} 0 & \text{if } a_n \ge 0 \\ a_n & \text{if } a_n < 0 \end{cases}$$

~~Theorem~~ 8.19

(2) Let $\sum a_n$ be a given series with real valued terms and define $p_n = \frac{|a_n| + a_n}{2}$ and $q_n = \frac{|a_n| - a_n}{2}$

Then, i) If $\sum a_n$ is conditionally convergent, both $\sum p_n$ and $\sum q_n$ diverge.

(ii) If $\sum |a_n|$ converge, both $\sum p_n$ and $\sum q_n$ converge and we have,

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} p_n - \sum_{n=1}^{\infty} q_n$$

Proof: part (i)

Hypothesis : $\sum a_n$ is conditionally convergent

: $\sum a_n$ converges but $\sum |a_n|$ diverges $\rightarrow$ (1)

Claim: Both $\sum p_n$ and $\sum q_n$ are divergent

We know that

$$2p_n = a_n + |a_n|$$

$$|a_n| = 2p_n - a_n$$

$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} 2p_n - \sum_{n=1}^{\infty} a_n$$

$$= 2\sum_{n=1}^{\infty} p_n - \sum_{n=1}^{\infty} a_n \longrightarrow (2)$$

To prove that : $\sum p_n$ is divergent

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If possible, assume that $\sum p_n$ is convergent

$2\sum p_n$ is convergent

already ① $\Rightarrow \sum a_n$ is convergent

$2\sum p_n - \sum a_n$ is convergent by ①

$\Rightarrow \sum |a_n|$ is convergent, which is contradicts.

$\therefore$ Our assumption is wrong. $\therefore \sum p_n$ is divergent

Similarly, we can show that $\sum q_n$ is also divergent

Hence part (i).

part(ii)

$\sum a_n$ is absolute convergent

$\therefore \sum a_n$ and $\sum |a_n|$ are convergent

$\Rightarrow \sum a_n + \sum |a_n|$ is convergent

$\Rightarrow \sum (a_n + |a_n|)$ is convergent

$\Rightarrow \sum 2p_n$ is convergent

$\Rightarrow 2\sum p_n$ is convergent

$\Rightarrow \sum p_n$ is convergent

IIIly $\sum q_n$ is convergent

By defn of 'p_n' and 'q_n'

p_n = positive terms and some terms of the alternating series and

q_n - negative terms and some terms of the alternating series.

$\sum_{n=1}^{\infty} a_n = a_1 - a_2 + a_3 - a_4 +$

$\sum a_n = \sum p_n + \sum q_n$ means

$= \sum (+a_n) + \sum (-a_n)$

$= \sum a_n - \sum a_n$ terms coming out of negative elements

$\therefore \sum a_n = \sum p_n - \sum a_n$ Hence (ii) and the theorem.

~~Hmm~~ Dirichlet's Test and Abel's Test

Theorem : 8.27

State and prove Abel's partial summation

Statement;

3 If $\{a_n\}$ and $\{b_n\}$ are two sequence of [illegible] numbers, define $A_n = a_1 + a_2 + \cdots + a_n$ then we have identity.

$$\sum_{k=1}^{n} a_k b_k = A_n b_{n+1} - \sum_{k=1}^{n} A_k (b_{k+1} - b_k)$$

therefore $\sum_{k=1}^{n} a_k b_k$ converges if both the series $\sum_{k=1}^{\infty} A_k (b_{k+1} - b_k)$ and the sequence $\{A_n b_{n+1}\}$ converge.

proof:

Hypothesis : $A_n = a_1 + a_2 + \cdots + a_n$ $(n = 1, 2, \ldots)$

Define, $A_0 = 0$

Consider, $\sum_{k=1}^{n} (A_k - A_{k-1}) b_k$

$$= (A_1 - A_0) b_1 + (A_2 - A_1) b_2 + \cdots + (A_n - A_{n-1}) b_n$$

$$= a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$$

$$= \sum_{k=1}^{n} a_k b_k \longrightarrow (1)$$

$A_1 = a_1$

$A_2 = a_1 + a_2$

$A_2 = A_1 + a_2$

$a_2 = A_2 - A_1$

In general

$a_n = A_n - A_{n-1}$

on the otherhand

$$\sum_{k=1}^{n} (A_k - A_{k-1}) b_k$$

$$= \sum_{k=1}^{n} A_k b_k - \sum_{k=1}^{n} A_{k-1} b_k \longrightarrow (2)$$

Consider $\sum_{k=1} A_k b_{k+1}$

$= A_1 b_2 + A_2 b_3 + \cdots + A_{n-1} b_n + A_n b_{n+1}$ Insert $A_0 b_1 = 0 \cdot b_0 = 0$

$= A_0 b_1 + A_1 b_2 + \cdots + A_{n-1} b_n + A_n b_{n+1}$ $(\because A_0 = 0)$

$$\sum_{k=1}^{n} A_k b_{k+1} = \sum_{k=1}^{n} A_{k-1} b_k + A_n b_{n+1}$$

$$\therefore \sum_{k=1}^{n} A_{k-1} b_k = \sum_{k=1}^{n} A_k b_{k-1} - A_n b_{n+1} \longrightarrow (3)$$

(Note that (3) gives an expression for the second term on the R.H.S of (2))

Substituting (3) in (2) we get,

$$\sum_{k=1}^{n} (A_k - A_{k-1}) b_k = \sum_{k=1}^{n} A_k b_k - \sum_{k=1}^{n} A_k b_{k+1} + A_n b_{n+1}$$

$$= \sum_{k=1}^{n} A_k (b_k - b_{k+1}) + A_n b_{n+1} \longrightarrow (4).$$

Note that the L.H.S of (1) & (4) are same.

Their R.H.S are equal.

(i.e) we have,

$$\sum_{k=1}^{n} a_k b_k = \sum_{k=1}^{n} A_k (b_k - b_{k+1}) + A_n b_{n+1} \longrightarrow (5)$$

Hence part (i)

Note that this identity is an expression for the n^{th} partial sum of the series $\sum_{n=1}^{\infty} a_n b_n$

This infinite series converges only if "the sequence of its partial sums given by the R.H.S of (5)" is convergent (or) $\sum_{n=1}^{\infty} a_n b_n$ is convergent. only if both the terms on the R.H.S converge when the series $\sum_{k=1}^{\infty} A_n(b_n - b_{n+1})$ and the sequence $\{A_n b_{n+1}\}_{n=1}^{\infty}$ are convergent $\sum_{k=1}^{\infty} A_n(b_{n+1} - b_n)$

Hence part (ii) and the theorem.

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